Last Time: - Set operations
- Basic Maths proofs w/ sets

Prop: Let A, B, C, and D be sets. We have

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

pf: Let A, B, C, and D be sets.

($\subseteq$): Let $x \in (A \times B) \cap (C \times D)$ be arbitrary. Thus $x \in A \times B$ and $x \in C \times D$ by definition of intersection. Thus we may write $x = (a, b)$ for some $a \in A$ and $b \in B$ because $x \in A \times B$. On the other hand, we know $x \in C \times D$, so $x = (c, d)$ for some $c \in C$ and $d \in D$. As $(a, b) = x = (c, d)$, we see $a = c$ and $b = d$. Hence $a = c \in C$ and $b = d \in D$. Thus $a \in A \cap C$ and $b \in B \cap D$ by definition of set intersection. Finally $x = (a, b) \in (A \cap C) \times (B \cap D)$ by definition of the Cartesian set product.

*($\supseteq$): Left as an exercise ...

NB: the above proof was done via the "two-containments" method for proving set equalities.

We can also use a string of set equalities
(compare with Algebra of Equivalences for propositions).

↳ Done by manipulating predicates in set-builder notation.

pf (Alternate proof): Let A, B, C, and D be sets. We have the following:

$$(A \times B) \cap (C \times D) = \{ x : x \in A \times B \text{ and } x \in C \times D \}$$ defn of $\cap$

$$= \{ x : \underbrace{x = (a,b) \text{ for some } a \in A \text{ and } b \in B}_{\text{and } x = (c,d) \text{ for some } c \in C \text{ and } d \in D} \}$$ defn of Cartesian product

$$= \{ x : \underbrace{x = (\varphi, \psi) \text{ where } \varphi \in A \text{ and } \psi \in B}_{\text{and } \varphi \in C \text{ and } \psi \in D} \}$$ Simplification of equal objects (x)

$$= \{ (\varphi, \psi) : (\varphi \in A \text{ and } \varphi \in C) \text{ and } \psi \in B \text{ and } \psi \in D \}$$ Commutativity

$$= \{ (\varphi, \psi) : \varphi \in A \cap C \text{ and } \psi \in B \cap D \}$$ defn of intersection

$$= (A \cap C) \times (B \cap D).$$ definition of Cartesian product

Hence we conclude $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ via the above string of set equalities. ///

Exercise: Re-prove the set equalities from the previous lecture using the method illustrated above.

Reminder of Summation Notation.

Suppose you have real numbers $a_1, a_2, \cdots, a_n$.

The sum of these is $a_1 + a_2 + \cdots + a_n$,

which can be abbreviated using "sigma-notation":

$$\sum_{k=1}^{n} a_k$$

sigma... → bounds on index. (n above, 1 below)

k is the index

a_k is the k^{th} term of sum.

Ex: We can abbreviate $1+2+3+\cdots+15 = \sum_{k=1}^{15} k$.

Ex: $1^2+2^2+3^2+\cdots+150^2 = \sum_{k=1}^{150} k^2$.

So 9 is the third term (b/c $k=3$ yields $3^2=9$)

Ex: $\frac{3(4)}{2} + \frac{4(5)}{2} + \cdots + \frac{n(n+1)}{2} = \sum_{k=3}^{n} \frac{k(k+1)}{2}$.

Mathematical Induction

Question: Suppose $P(n)$ is a predicate with domain the natural numbers.

How do I prove $\forall n \in \mathbb{N}\,[P(n)]$?

↳ So far, we know we could try a "direct proof"...

What about: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$?

we need Proof by Induction!

IDEA: Statement $P(k)$ is like the k^{th} domino in a domino chain

If we prove $P(k)$ is true, that means the k^{th} domino falls over.

So proving ① the first domino falls and ② the dominoes are close enough to knock the next over, then every domino falls!

Formally: To prove $\forall n \in \mathbb{N}\,[P(n)]$: $\mathbb{N} = \{0, 1, \dots\}$

① Verify $P(0)$ (could be anything.) (Base Case)

② Prove $\forall_{k \in \mathbb{N}}[P(k) \to P(k+1)]$ (Inductive Step)

③ Conclude $\forall n \in \mathbb{N}\,[P(n)]$ ✱ ☺.

Prop: For all $n \in \mathbb{N}$ we have $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. ✱

pf: We proceed by induction on $n \in \mathbb{N}$.

Base Case: When $n = 0$ we have $\sum_{k=1}^{n} k = 0$ because we made an empty sum. On the other hand, $\frac{0(0+1)}{2} = 0$ by basic arithmetic. So $\sum_{k=1}^{0} k = 0 = \frac{0(0+1)}{2}$.

Hence the formula holds when $n = 0$.

$\forall m \in \mathbb{N}[P(m) \to P(m+1)]$

Inductive Step: Suppose the formula holds for some arbitrary $m \in \mathbb{N}$. Thus we have (assumed $P(m)$.)

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}.$$

(WTS: $P(m+1)$ is true

i.e. $\sum_{k=1}^{m+1} k = \frac{(m+1)((m+1)+1)}{2}$

i.e. $\sum_{k=1}^{m+1} k = \frac{(m+1)(m+2)}{2}$)

We compute the following

$$\sum_{k=1}^{m+1} k = \left(\sum_{k=1}^{m} k\right) + (m+1)$$

(last term)

$$= \frac{m(m+1)}{2} + (m+1) = (m+1)\left(\frac{m}{2} + 1\right) = (m+1)\left(\frac{m+2}{2}\right)$$

$$= \frac{(m+1)(m+2)}{2}.$$

Hence the formula holds for $m+1$ as well, so the inductive step holds.

Hence we conclude the original statement holds via the principle of Mathematical Induction. ▯

Exercise: Try to prove ① $\sum_{k=1}^{n} 1 = n$ for all $n \in \mathbb{N}$.

② $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ for all integers $n \geq 1$.

Next time: Fibonacci Numbers + More induction.

$F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

is the recursively-defined sequence of Fibonacci Numbers.

→ More cool stuff next time.